and

$$\Delta(f) = |H(f)|^2 |H(f+1/2)|^2 [S_x(f) - S_x(f+1/2)]^2.$$

Further, the inequality is strict if $\Delta(f) > 0$ on a subset of $[-1/4, 1/4]$ of positive measure.

Comments:

1) If ideal "brickwall" filters are used, then $\Delta(f)$ is zero (except possibly for a finite number of points), and

$$\exp\left\{\int_{-1/4}^{1/4} \log_e S_x(f) S_x(f+1/2u)\, df\right\} = \sigma_x^2 \gamma_x^2.$$

Hence, (8) and (1) are identical and nothing is lost, in a rate-distortion sense, by separately encoding the subbands.

2) If $x(n)$ is white, then $\Delta(f) = 0$, and again (8) is equal to (1). More generally, if $S_x(f)$ is symmetric about $f = 1/4$, then $\Delta(f) = 0$ and (8) is equal to (1).

3) Since $\log_e(\cdot)$ is a strictly increasing function, if $\Delta(f) > 0$ on a subset of $[-1/4, 1/4]$ of positive measure, then the left hand side of (9) is strictly larger than $\sigma_x^2 \gamma_x^2$, and so the subband coding must be suboptimum in a rate-distortion sense compared to encoding the source directly.

4) For a given source and filters, numerical evaluation of (9) provides a straightforward way of determining the rate-distortion consequences of using a particular filter (for example, a filter with very short kernel length).

III. Proof of Theorem

The proof of the theorem is as follows. Begin with

$$2\sqrt{\sigma_d^2 \gamma_d^2 \sigma_s^2 \gamma_s^2} = \exp\left\{\int_{-1/4}^{1/4} \log_e 4\, df\right\} \exp\left\{\frac{1}{2}\int_{-1/2}^{1/2} \log_e S_d(f)\, df\right\} \cdot \exp\left\{\frac{1}{2}\int_{-1/2}^{1/2} \log_e S_s(f)\, df\right\}.$$

Substituting from (3), (4), and manipulating yields

$$2\sqrt{\sigma_d^2 \gamma_d^2 \sigma_s^2 \gamma_s^2} = \exp\left\{\int_{-1/4}^{1/4} \log_e\left[\left(|G(f)|^2 S_x(f) + |G(f+1/2)|^2 S_x(f+1/2)\right) \cdot \left(|H(f)|^2 S_x(f) + |H(f+1/2)|^2 S_x(f+1/2)\right)\right] df\right\}. \quad (10)$$

From (5), (6) it follows that

$$|G(f)|^2 = |H(f+1/2)|^2$$

and

$$|H(f+1/2)|^2 = 1 - |H(f)|^2.$$

substituting these relations into (10) yields, upon simplification, the left-hand side of (9). Since $\log_e(\cdot)$ is a strictly increasing function for positive argument, it follows that

$$\begin{aligned} 2\sqrt{\sigma_d^2 \gamma_d^2 \sigma_s^2 \gamma_s^2} &= \exp\left\{\int_{-1/4}^{1/4} \log_e\left[\Delta(f) + S_x(f) S_x(f+1/2)\right] df\right\} \\ &\geq \exp\left\{\int_{-1/4}^{1/4} \log_e\left[S_x(f) S_x(f+1/2)\right] df\right\} \\ &= \exp\left\{\int_{-1/2}^{1/2} \log_e S_x(f)\, df\right\} \\ &= \sigma_x^2 \gamma_x^2. \end{aligned}$$

Acknowledgment

The author wishes to thank an anonymous reviewer for pointing out that comment 2 applies for symmetric spectral densities.

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On Universal Quantization by Randomized Uniform/Lattice Quantizers

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***Abstract*—Uniform quantization with dither, or lattice quantization with dither in the vector case, followed by a universal lossless source encoder (entropy coder), is a simple procedure for universal coding with**

Manuscript received April 9, 1990; revised May 30, 1991. M. Feder was supported by The Andrew W. Mellon Foundation, Woods Hole Oceanographic Institution.

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IEEE Log Number 9103969.

distortion of a source that may take continuously many values. The rate of this universal coding scheme is examined, and a general expression is derived for it. An upper bound for the redundancy of this scheme, defined as the difference between its rate and the minimal possible rate, given by the rate distortion function of the source, is derived. This bound holds for all distortion levels. Furthermore, a composite upper bound on the redundancy as a function of the quantizer resolution that leads to a tighter bound in the high rate (low distortion) case is presented.

Index Terms-**Uniform and lattice quantization, randomized quantization, universal coding, rate-distortion performance.**

I. Introduction and Summary of Results

A. The Dithered Uniform/Lattice Quantizer

Uniform quantization with dither, or more generally lattice quantization with dither, followed by a universal lossless source encoder (entropy encoder) is a simple procedure for universal coding with distortion of a source that may take continuous real values. This procedure is universal since it does not depend on the source statistics. Due to the dither, the distortion in this procedure is independent of the source value. In this correspondence, we consider the rate performance, i.e., the entropy, of the uniform (lattice) randomized quantizer as compared with the optimal rate given by the rate-distortion function of the source.

The uniform quantizer with dither, or the randomized quantizer, is defined as follows. The code points of the uniform quantizer are $\{0, \pm\Delta, \pm 2\Delta, \cdots\}$. The quantizer function $Q: \mathscr{R} \to \mathscr{R}$ is such that

$$Q(x) = i\Delta, \qquad \text{for } i\Delta - \Delta/2 \le x \le i\Delta + \Delta/2.$$

Let Z be a random variable distributed uniformly in the interval $[-\Delta/2, \Delta/2]$. The universal quantizer with dither represents a source value x as

$$u = Q(x+z) - z \tag{1}$$

where z is a sample of Z. It is easy to show that for any difference distortion measure $\delta(x-u)$, e.g., a distortion measure of the form $|x-u|^r$, the average error of this quantizer is independent of x, i.e.,

$$E_z\{\delta(Q(x+z) - z - x)\} = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} \delta(\alpha)\,d\alpha = \epsilon. \tag{2}$$

The generalization of the uniform quantizer to the vector case is the lattice quantizer. The code points of a K-dimensional lattice quantizer form a K-dimensional lattice $\{L_K\}$. The quantizer $Q_K(\cdot)$ maps every vector $\boldsymbol{x} \in \mathscr{R}^K$ into the nearest lattice point $\boldsymbol{l}_i \in L_K$. For a difference distortion measure, the region of all K-vectors mapped into a lattice point $\boldsymbol{l}_i \in L_K$ is the Voronoi region,

$$\mathscr{V}(\boldsymbol{l}_i) = \{\boldsymbol{x} \in R^K: \delta(\boldsymbol{x} - \boldsymbol{l}_i) \le \delta(\boldsymbol{x} - \boldsymbol{l}_j) \text{ for all } j \ne i\}.$$

Clearly, the uniform scalar quantizer is the special case, Q_1, of the lattice quantizer. Now let $\boldsymbol{Z}$ be a K-dimensional random variable uniformly distributed over the basic cell of the lattice, $\mathscr{V}_0 = \mathscr{V}(\boldsymbol{0})$, the Voronoi region of the lattice point $\boldsymbol{0}$. The lattice quantizer with dither represents a source vector $\boldsymbol{x} \in \mathscr{R}^K$ by a vector,

$$\boldsymbol{u} = Q_K(\boldsymbol{x} + \boldsymbol{z}) - \boldsymbol{z}, \qquad \boldsymbol{u} \in \mathscr{R}^K, \tag{3}$$

where $\boldsymbol{z}$ is a sample from the random vector $\boldsymbol{Z}$. As for the scalar case, one can easily show that for a difference distortion measure the average distortion is independent of the value $\boldsymbol{x}$; thus, only difference distortion measures will be considered.

The quantizers above can be used to encode a source vector $\boldsymbol{x} \in \mathscr{R}^n$ as follows. In the scalar case, a dither is added independently to each source component and the result is then quantized component by component. In the lattice case, we assume that K divides n and the input is considered as a concatenation of n/K K-dimensional vectors, quantized independently, using independent vector samples of the dither. In both cases the entropy coder will then take into account the statistical properties of the entire n-dimensional vector.

B. Previous Results

In [1] and [2], the rate of these quantizers, in encoding a general n-dimensional source, was compared to the rate achieved by the optimal entropy-constrained vector quantizer (ECVQ). (This optimal n-dimensional ECVQ can be designed, e.g., using the technique of [3].) It was shown, for example, that for a block of length n, emitted from any source, X,

$$\begin{aligned}\frac{1}{n}H(Q_1(X+Z) - Z \mid Z) &= \frac{1}{n}H(Q_1(X+Z) \mid Z) \\ &\le \frac{1}{n}H(Q^n_{\text{opt}}(\epsilon; X)) + \frac{1}{2}\log\frac{4\pi e}{12},\end{aligned} \tag{4}$$

where $Q_1(\cdot)$ is the randomized scalar quantizer having a step size Δ, $\epsilon = \Delta^2/12$ is its mean square error, $Q^n_{\text{opt}}(\epsilon; X)$ is the optimal n-dimensional ECVQ for the source X having a mean-square error ϵ, and $H(\cdot)$ is the entropy function that may denote here either a regular entropy or a differential entropy. The results in [1] and [2] were derived for a slightly different realization of the dither vector where the same sample Z is repeated n times. Throughout the correspondence, the logarithm base is 2 and the entropy is measured in bits. Note that $1/2 \log 4\pi e/12 \approx 0.754$ bits. For notational short-hand we use $H(Q_1 \mid Z)$ for $H(Q_1(X+Z) \mid Z)$ and $H(Q^n_{\text{opt}})$ for $H(Q^n_{\text{opt}}(\epsilon; X))$.

In [1], the bound (4) has also been generalized to the case of a lattice quantizer with dither and a mean-square error distortion measure, to yield the bound

$$\frac{1}{n}H(Q_K \mid Z) \le \frac{1}{n}H(Q^n_{\text{opt}}) + \frac{1}{2}\log 4\pi e G_K, \tag{5}$$

where G_K is the generalized second moment of the lattice, see [4], i.e.,

$$G_K = \frac{1}{K}\frac{\int_{\mathscr{V}_0} \|\boldsymbol{x} - \hat{\boldsymbol{x}}\|^2\,d\boldsymbol{x}}{\left(\int_{\mathscr{V}_0} d\boldsymbol{x}\right)^{1+2/K}} \quad \boldsymbol{x}, \hat{\boldsymbol{x}} \in \mathscr{R}^K, \tag{6}$$

and $\hat{\boldsymbol{x}}$ is the centroid of the polytope $\mathscr{V}_0$. Note that $G_1 = 1/12$, and the minimum value of $G_K \to 1/2\pi e$ as $K \to \infty$ so that the minimum value of $1/2 \log 4\pi e G_K \to 0.5$.

In [2], bounds similar to (5) have been derived for other distortion criteria.

C. Summary of New Results

The main observation made in this work is that the rate of the randomized uniform/lattice quantizer can be written as the mutual information between the input and the output of the additive noise channel shown in Fig. 1. The input to this auxiliary channel is the source to be quantized and the additive noise is uniformly distributed over the mirror image of the quantizer basic cell, i.e., as the dither if the cell is symmetric. Specifically, denoting the additive noise by N, where its p.d.f. $f_N(n) = f_Z(-n)$, the source input vector by X and the output by $Y = X + N$, then,

$$H(Q \mid Z) = I(X; Y) = H(X + N) - H(N). \tag{7}$$

As will be shown, this general result allows us to calculate universal upper bounds on the rate of the randomized quantizers.

The interesting quantity for universal coding with distortion is the excess rate over the rate-distortion function of that coding scheme. We thus define the redundancy of the randomized lattice quantizer, of dimension K and at the distortion level ϵ, as

$$\rho_{K,n}(\epsilon) = \frac{1}{n} H(Q_K \mid Z) - R_n(\epsilon), \tag{8}$$

where $R_n(\epsilon)$ is the rate distortion function of an n-dimensional random vector

$$R_n(\epsilon) = \inf_{\{f_{U/X}(u/x):(1/n)\bar{\delta} \le \epsilon\}} \frac{1}{n} I(X; U) \tag{9}$$

and where $f_{U/X}(u/x)$ is a conditional p.d.f. of the representation u given the source x, the term $(1/n)\bar{\delta} = (1/n)E_{u,x}\{\delta(u - x)\}$ is the average distortion per source symbol between the source and its representation, and $I(X; U) = I(X_1, \cdots, X_n; U_1, \cdots, U_n)$ is the mutual information between the random vectors X and U.

From the observation (7) it will be shown that

$$\rho_{K,n}(\epsilon) \le C, \tag{10}$$

where $C = C(\delta(\cdot), K)$ is the *capacity* of the channel of Fig. 1 under the constraint that the input X satisfies

$$\frac{1}{n} E\{\delta(X)\} \le \epsilon.$$

This upper bound is true for all sources and for all distortion levels. For a square-error distortion measure this capacity, at dimension K, can be bounded by

$$C < \frac{1}{2} \log 4\pi e G_K. \tag{11}$$

Now, note that the RHS of (11) is the same value as in [1]; however, we actually improve upon the result of [1] since the inequality (11) is strict and since we have bounded the difference between the rate of the universal quantizer and the rate-distortion function, while in [1] the upper bound was for the excess rate as compared to the optimal n-dimensional quantizer. We will also observe that our upper bound is reachable and thus it cannot be improved by a constant bound that holds for all distortion levels. As will be seen the bound is reached in a case that can be describes as "low resolution quantization."

Another new result that follows from (7) is an upper bound on the redundancy $\rho_{K,n}(\epsilon)$ as a function of the distortion ϵ. We were motivated to derive this bound by the availability of upper bound, tighter than (10) and (11), for deterministic uniform and lattice

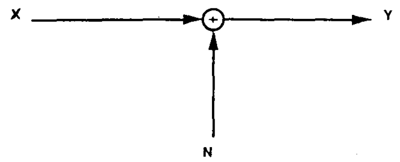

Fig. 1. Auxiliary channel. $N \sim \mathcal{U}(-\Delta/2, \Delta/2)$ for scalar uniform quantizer. $N \sim \mathcal{U}(V_0^-)$ for vector lattice quantizer.

quantizers, in cases which can be characterized as "high resolution," see [5] and [6]. For a square-error distortion measure, this bound on the redundancy can be written as

$$\rho_{K,n}(\epsilon) \le \frac{1}{2} \log 2\pi e G_K + \frac{1}{2} \log \frac{P(X+N)}{P(X)}, \tag{12}$$

where the first term is the bound of [5] and [6], which holds for deterministic quantizers under the "high resolution" assumption made there, and the second term involves the entropy power $P(X)$ of the source, and the entropy power $P(X + N)$ of the source with the additive noise of Fig. 1. Note that at high resolution, $P(X + N) \approx P(X)$, and thus the upper bound of [5] and [6], proved there for deterministic quantizers, holds for randomized quantizers as well, i.e.,

$$\rho_{K,n}(\epsilon) \le \frac{1}{2} \log 2\pi e G_K, \qquad \text{as } \epsilon \to 0. \tag{13}$$

II. Detailed Presentation and Derivation

A. An Expression for the Rate of the Randomized Uniform / Lattice Quantizer

The general expression for the rate is initially presented, in Theorem 1, for the scalar case, and it is then extended to the vector case and lattice quantizers.

Theorem 1: The entropy of the randomized uniform quantizer, with step-size Δ, in encoding the random variable X, is given by

$$\begin{aligned} H(Q \mid Z) &= I(X;Y) = H(X+N) - H(N) \\ &= H(Y) - \log \Delta, \end{aligned} \tag{14}$$

where $Y = X + N$, X is the source, N is a random variable independent of X distributed uniformly over $[-\Delta/2, \Delta/2]$, and $I(X; Y)$ is the mutual information between X and Y.

Recall that $H(Q \mid Z) = H(Q(X + Z) - Z \mid Z) = H(Q(X + Z) \mid Z)$, and we are interested in this conditional entropy since in a randomized quantization procedure the dither is a pseudo-noise known to the receiver.

Proof: The quantization levels of the uniform quantizer are denoted,

$$q_i = i \cdot \Delta, \qquad i = 0, \pm 1, \pm 2, \cdots.$$

The dither $Z \sim \mathcal{U}[-\Delta/2, \Delta/2]$. For any sample dither value z

the quantizer output can take one of the values

$$q_i' = Q(x+z) - z = q_i - z, \qquad i = 0, \pm 1, \pm 2, \cdots,$$

and the probability to get such a value, q_i', given z, is

$$\begin{aligned} P(q_i' \mid z) &= \text{Prob}\{q_i - \Delta/2 \le X + z \le q_i + \Delta/2\} \\ &= \text{Prob}\{q_i' - \Delta/2 \le X \le q_i' + \Delta/2\} \\ &= \int_{q_i'-\Delta/2}^{q_i'+\Delta/2} f_X(x)\, dx. \end{aligned}$$

Now we observe that the p.d.f. of the r.v. Y is given by

$$\begin{aligned} f_Y(y) = f_X(\cdot) * f_N(\cdot) &= \int_{-\infty}^{\infty} f_X(x) f_N(y-x)\, dx \\ &= \frac{1}{\Delta} \int_{y-\Delta/2}^{y+\Delta/2} f_X(x)\, dx, \end{aligned}$$

where $*$ denotes the convolution operator. Thus, we can write

$$P(q_i' \mid z) = \Delta \cdot f_Y(q_i') = \Delta \cdot f_Y(q_i - z).$$

The entropy of the quantizer output, conditioned on the dither, is given explicitly by

$$\begin{aligned} H(Q \mid Z) &= E_z\left\{ -\sum_{i=-\infty}^{\infty} P(q_i' \mid z) \log P(q_i' \mid z) \right\} \\ &= E_z\left\{ -\sum_{i=-\infty}^{\infty} \Delta \cdot f_Y(q_i - z) \log \Delta \cdot f_Y(q_i - z) \right\}. \end{aligned} \tag{15}$$

Using the dither p.d.f., the expectation in (15) can be written explicitly as

$$\begin{aligned} H(Q \mid Z) &= \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} \left[-\sum_{i=-\infty}^{\infty} \Delta \cdot f_Y(q_i - z) \log f_Y(q_i - z) \right] \cdot dz - \log \Delta \\ &\text{(the summation and integration can be complemented to a continuous integral)} \\ &= -\int_{-\infty}^{\infty} f_Y(y) \log f_Y(y)\, dy - \log \Delta \\ &= H(Y) - \log \Delta, \end{aligned} \tag{16}$$

and this is the mutual information between the input and output of the channel in Fig. 1. □

The expression for the rate of the quantizer, as provided by this theorem, depends on the distortion via the uniform quantizer step size Δ. For any given difference distortion measure $\delta(\cdot)$, we can use the relation

$$\epsilon = E\{\delta(Z)\} = \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} \delta(\alpha)\, d\alpha = \phi(\Delta) \tag{17}$$

and get, by substituting $\Delta = \theta^{-1}(\epsilon)$, the rate as a function of the average distortion ϵ.

When we consider a vector source $X = X_1 \cdots X_n$ but still use a scalar uniform quantizer and i.i.d. dither values $Z = Z_1 \cdots Z_n$, it is easy to show by a straight-forward extension of Theorem 1 that the rate per symbol of the scalar randomized quantizer, denoted $H(Q_1 \mid Z)$, is

$$\frac{1}{n} H(Q_1 \mid Z) = \frac{1}{n} I(X; Y) = \frac{1}{n} H(Y) - \log \Delta, \tag{18}$$

where $Y = X + N$ is the output of the auxiliary channel with input X. The quantizer step size now defines the distortion per symbol ϵ via

$$\epsilon = \frac{1}{n \cdot \Delta^n} \int_{D^n} \delta(\alpha)\, d\alpha = \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} \delta(\alpha_i)\, d\alpha_i, \tag{19}$$

where D^n is the n-dimensional cube $\{u \in R^n: -\Delta/2 \le u_i \le \Delta/2\}$ and where the last equality holds when the distortion measure satisfies $\delta(x - u) = \sum_{i=1}^{n} \delta(x_i - u_i)$, i.e., it is a single symbol distortion measure.

The most general case we consider is the quantization of an n-dimensional vector source using a randomized K-dimensional lattice quantization. As noted before, we assume that K divides n; the source vector is divided into n/K vectors of dimension K and each subvector is coded independently. The randomized lattice quantization is performed by first adding a dither Z_K to each source subvector X_K, of dimension K, and then representing the result by the nearest lattice point. The dither vector is uniformly distributed over the lattice Voronoi cell $\mathscr{V}_0$.

Let N_K be a random K-dimensional vector distributed uniformly over $\mathscr{V}_0^-$, where $\mathscr{V}_0^- = \{x: -x \in \mathscr{V}_0\}$ is the reflection of $\mathscr{V}_0$. Let Z, X, and N be concatenations of the n/k subvectors Z_K, X_K and N_K, and let $Y = X + N$. Then the rate (per symbol) of the randomized lattice quantizer in encoding the entire n-dimensional source vector X is

$$\frac{1}{n} H(Q_K \mid Z) = \frac{1}{n} I(X; Y) = \frac{1}{n} H(Y) - \frac{1}{K} \log V, \tag{20}$$

where V is the volume of the K-dimensional Voronoi cell of the lattice. This generalization of Theorem 1 is shown in Appendix A.

As in the scalar case, the average distortion per symbol is expressed in terms of the volume V,

$$\epsilon = \frac{1}{n \cdot V^{n/K}} \int_{(\mathscr{V}_0)^{n/K}} \delta(n)\, dn = \frac{1}{K \cdot V} \int_{\mathscr{V}_0} \delta(n_K)\, dn_K, \tag{21}$$

where $(\mathscr{V}_0)^{n/K}$ is the n/K-fold cartesian product of the cell $\mathscr{V}_0$, and the last equality holds when the distortion measure can be expressed as a sum of K-dimensional distortion measures.

B. Universal Upper Bound on the Redundancy of Randomized Quantizers

The general expression for the randomized quantizer entropy, presented above, does not provide an insight to its performance with respect to the optimal performance. In this section we derive a *universal* upper bound for the difference between this entropy and the rate distortion function, i.e., for the redundancy of the randomized quantizer. We start again with the scalar case.

Theorem 2: The redundancy of the scalar uniform randomized quantizer satisfies

$$\rho_{1,1}(\epsilon) = H(Q \mid Z) - R(\epsilon) \le C, \tag{22}$$

where $\epsilon = 1/\Delta \int_{-\Delta/2}^{\Delta/2} \delta(\alpha)$, $\delta(\cdot)$ is the distortion measure, Δ is the quantizer step size, $R(\epsilon)$ is the rate-distortion function, and C is the constrained capacity of the channel of Fig. 1,

$$\begin{aligned} C &= \sup_{\{f_X(x):\, E\{\delta(X)\} \le \epsilon\}} I(X; X+N) \\ &= \sup_{\{f_X(x):\, \int \delta(x) f_X(x)\, dx \le \epsilon\}} I(X; X+N). \end{aligned} \tag{23}$$

Proof: The rate-distortion function is the minimum of $I(X;U)$ over all U satisfying

$$E\{\delta(X-U)\} \le \epsilon, \tag{24}$$

where, since the minimization is over $f_{U/X}(u/x)$, the r.v. U achieving the rate-distortion function may be chosen to be independent of N.

From Theorem 1, $H(Q \mid Z) = I(X;Y)$; thus, we can write

$$H(Q \mid Z) - I(X;U) = I(X;Y) - I(X;U), \tag{25}$$

where we recall that $Y = X + N$ is the output of the auxiliary channel. Using the relations

$$\begin{aligned} I(X;Y) - I(X;U) &= H(X) - H(X \mid Y) - \{H(X) - H(X \mid U)\} \\ &= H(X \mid U) - H(X \mid Y), \end{aligned}$$

and

$$\begin{aligned} I(X;Y \mid U) - I(X;U \mid Y) &= H(X \mid U) - H(X \mid Y, U) \\ &\quad - \{H(X \mid Y) - H(X \mid U, Y)\} \\ &= H(X \mid U) - H(X \mid Y), \end{aligned}$$

we get that

$$\begin{aligned} I(X;Y) - I(X;U) &= I(X;Y \mid U) - I(X;U \mid Y) \\ &\le I(X;Y \mid U). \end{aligned} \tag{26}$$

Since U may be chosen independent of N,

$$\begin{aligned} I(X;Y \mid U) &= H(Y \mid U) - H(N) = H(Y-U \mid U) - H(N) \\ &\le H(Y-U) - H(N) = I(X-U; Y-U). \end{aligned} \tag{27}$$

Using (24) and recalling the definition of C, we get

$$I(X-U; Y-U) \le C. \tag{28}$$

Combining (25)–(28), we get $H(Q \mid Z) - I(X;U) \le C$. This holds for any U satisfying (24) including U that achieves the rate-distortion function, and so the proof is complete. □

Let X^* be the random variable achieving the capacity in (23). It is easy to see that when X^* is encoded by the randomized quantizer, with distortion ϵ, its rate achieves the upper bound of (22). Specifically, since $E\{\delta(X^*)\} \le \epsilon$, the rate distortion of X^* for distortion ϵ is *zero*, (we can choose $U \equiv 0$, which satisfies the constraint, to achieve the rate distortion function). Since $H(Q(X^* + Z) \mid Z) = I(X^*; Y)$ where $Y = X^* + N$, and since X^* achieves the capacity.

$$\begin{aligned} H(Q(X^*+Z) \mid Z) - R(\epsilon; X^*) &= H(Q(X^*+Z) \mid Z) \\ &= I(X^*; X^*+N) = C. \end{aligned} \tag{29}$$

We conclude, then, that the bound on the redundancy is reachable and so it cannot be improved by any bound that holds for all distortion levels.

In Appendix B, the capacity (23) is investigated further. It is shown there that for a square-error distortion measure, this capacity can be bounded by

$$\frac{1}{2}\log\left(1 + \frac{2\pi e}{12}\right) < C < \frac{1}{2}\log\frac{4\pi e}{12}, \tag{30}$$

where the upper bound is achieved if the channel output, Y, is Gaussian. This bound, $(1/2)\log(4\pi e/12) \approx 0.754$ b, is the same as the bound calculated in [1] for the difference between the entropy of the randomized quantizer and the optimal ECVQ quantizer. We note, however, that when summing two independent random variables one gets a Gaussian r.v. only if these random variables are Gaussian. Thus, the capacity and so our bound is strictly smaller then $(1/2)\log(4\pi e/12)$.

Appendix B also contains upper and lower bounds for this channel capacity for distortion measures of the form $|x - u|^r$.

Theorem 2 can be generalized straightforwardly to the case of an n-dimensional source and K-dimensional lattice (where K divides n). Specifically, we get

$$\rho_{K,n}(\epsilon) = \frac{1}{n}H(Q_K \mid Z) - R_n(\epsilon) \le C, \tag{31}$$

where Q_K and $R_n(\epsilon)$ have been defined above, and where now C is the constrained capacity of the additive noise channel,

$$C = \sup_{\{f_X(x):\, (1/n)\int \delta(x) f_X(x)\, dx \le \epsilon\}} \frac{1}{n} I(X; X+N). \tag{32}$$

Note that for a general n-dimensional source and a uniform (scalar) quantizer, and for a single symbol distortion measure, the upper bound (the capacity) is the same as (23).

The bound in (31) is tight in the sense that we can find a source for which the difference between the quantizer rate and the rate distortion function is exactly C. As in the scalar case this is the source that achieves the capacity (32).

For a square-error distortion measure, $\delta(\boldsymbol{x} - \boldsymbol{u}) = \|\boldsymbol{x} - \boldsymbol{u}\|^2$, our universal bound is the regular power-constrained capacity which satisfies

$$\frac{1}{2}\log(1 + 2\pi e G_K) < C < \frac{1}{2}\log 4\pi e G_K. \tag{33}$$

Now, notice that as $K \to \infty$ the minimum value of G_K, i.e., the optimal second moment at dimension K, approaches $1/2\pi e$ (see e.g., [4]). Thus, as $K \to \infty$, the upper and lower bounds on the capacity approach each other and approach 0.5.

C. Bound on the Redundancy at Various Distortion Levels

Deterministic, uniform, and lattice quantizers have been investigated by Gish and Pierce [5], and by Gersho [6], [7]. Unlike our results for the randomized quantizer, the rate, or the entropy, of the deterministic quantizers has been calculated only in the high resolution (low distortion) case, and for sources with a smooth probability density function. Nevertheless, under these conditions it was shown that the optimal, entropy-constrained, quantizer becomes a uniform (or lattice) quantizer and, in general, the rate of the deterministic lattice quantizer of dimension K satisfies

$$\frac{1}{n}H(Q_K(X)) - R_n(\epsilon) \le \mu_K(\epsilon), \tag{34}$$

where, for a square-error distortion,

$$\lim_{\epsilon\to 0}\mu_K(\epsilon) = \frac{1}{2}\log 2\pi e G_K. \tag{35}$$

Note that this high resolution bound is better by 0.5 bit/sample then our universal bound!

We have derived an alternative upper bound on the redundancy that depends on the quantizer distortion. To simplify the exposition, we present the result in Theorem 3 for the square error distortion criterion. An extension to any other difference distortion measures is straight-forward.

Theorem 3: The redundancy of the randomized lattice quantizer, for a square error distortion measure, satisfies

$$\rho_{K,n}(\epsilon) = \frac{1}{n}H(Q_K \mid Z) - R_n(\epsilon) \le \frac{1}{2}\log\frac{P(Y)}{P(X)} + \frac{1}{2}\log 2\pi e G_K, \tag{36}$$

where X is the n-dimensional source to be quantized, $Y = X + N$ as above, and $P(\cdot)$ is the entropy power of the corresponding random vector.

Proof: Following the result in (20)

$$\frac{1}{n}H(Q_K \mid Z) - R_n(\epsilon) = \frac{1}{n}H(Y) - \frac{1}{K}\log V, \tag{37}$$

where V is the volume of the lattice's Voronoi cell. Since for a square-error distortion measure $\epsilon = (1/K)\int_{\mathscr{V}_0}(1/V)\|\alpha\|^2\, d\alpha = (1/K)\int_{\mathscr{V}_0}(1/V)\|\alpha\|^2\, d\alpha$, we get from (6) that $V^{2/K} = \epsilon/G_K$. Thus,

$$\frac{1}{K}\log V = \frac{1}{2}\log\frac{\epsilon}{G_K}. \tag{38}$$

Let $R_n^L(\epsilon)$ be Shannon's lower bound on the rate-distortion function. We can write,

$$R_n(\epsilon) \ge R_n^L(\epsilon) = \frac{1}{n}H(X) - \frac{1}{2}\log 2\pi e\epsilon. \tag{39}$$

Using (37)–(39), we get

$$\frac{1}{n}H(Q_K \mid Z) - R_n(\epsilon) \le \frac{1}{n}[H(Y) - H(X)] + \frac{1}{2}\log 2\pi e G_K. \tag{40}$$

Using the definition of the entropy power, e.g., $P(Y) = (1/2\pi e)2^{(2/n)H(Y)}$, we get (36). □

The quantity $n^{-1}[H(Y) - H(X)] = 1/2\log(P(Y)/P(X))$ can be a measure of the quantizer resolution; it tends to zero for high resolution quantization and smooth source density.

Consider, indeed, the high resolution case in details. In this case, we assume that for a given small ϵ_r, the quantizer is fine enough and the source p.d.f. smooth enough so that for each $\xi \in \mathscr{V}_0$,

$$f_X(\alpha) \approx f_X(\alpha + \xi) \text{ or } \frac{|f_X(\alpha) - f_X(\alpha+\xi)|}{|f_X(\alpha)|} < \epsilon_r. \tag{41}$$

Since $f_Y(\alpha)$ is the average over all $\xi \in \mathscr{V}_0$ of $f_X(\alpha + \xi)$, we get

$$f_Y(\alpha) \approx f_X(\alpha) \text{ or } \frac{|f_Y(\alpha) - f_X(\alpha)|}{|f_X(\alpha)|} < \epsilon_r. \tag{42}$$

It is easy to show, by simple algebraic manipulations, that if (42) holds for almost all α (besides, maybe a set of measure zero), then $|H(Y) - H(X)| < \epsilon_r[H(X) + \log M]$, where M is an upper bound for $f_X(x)$. This implies,

$$\frac{1}{n}[H(Y) - H(X)] \approx 0. \tag{43}$$

Thus, from (40), the redundancy of the lattice quantizer for the square-error distortion will be,

$$\lim_{\epsilon_r\to 0}\left[\frac{1}{n}H(Q \mid Z) - R_n(\epsilon)\right] \le \frac{1}{2}\log 2\pi e G_K = \lim_{\epsilon_r\to 0}\mu_K(\epsilon_r), \tag{44}$$

which is the redundancy for the deterministic quantizer, (35). Note that since $H(X) \approx H(Y)$, $R_n^L(\epsilon)$ approaches $R_n(\epsilon)$ and, thus, this upper bound, at high resolution, is universally achievable.

III. Examples

The rate of the uniform/lattice randomized quantizer, can now be easily assessed, for practical usage, using the expressions and the bounds derived in this correspondence.

We have tested a few examples where we have used the square error distortion measure. In the first example, we considered the rate of the quantizer when it operates on a Gaussian, memoryless, source. The rate can be determined by (14), which gives an expression for the exact rate, or from the bound (36). We note, however, that for the Gaussian source, both expressions are identical. In Fig. 2, we have plotted this rate as a function of the distortion, and compared it to the rate-distortion function of the Gaussian source (the straight line in the linear-log scale). Note that this rate is greater by at most 0.75 bits than the rate-distortion function, at the low resolution case. This is slightly less than the bound 0.754 since the Gaussian distribution is not the one that achieves the capacity. This result implies, however, that the capacity bound cannot be much tighter than 0.754!

The rate of the lattice quantizer as a function of the distortion, for encoding this memoryless Gaussian source, is depicted in Fig. 3, where the rate is compared to the randomized scalar quantizer to see the vector advantage. The rate was again computed via (36), where we have assumed that $G_K = 1/2\pi e$, i.e., the excess rate over

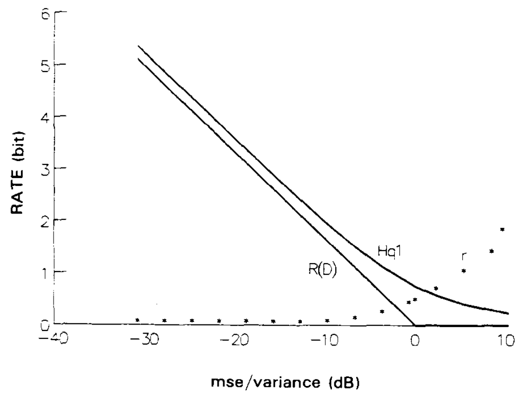

Fig. 2. Rate of the uniform randomized quantizer for a Gaussian source as compared to the rate-distortion function.

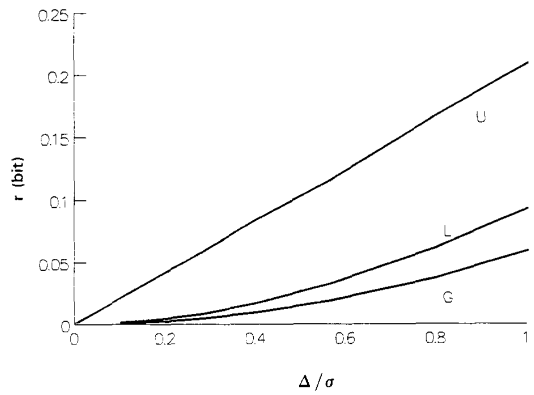

Fig. 4. Resolution measure for uniform, Laplacian, and Gaussian memoryless sources.

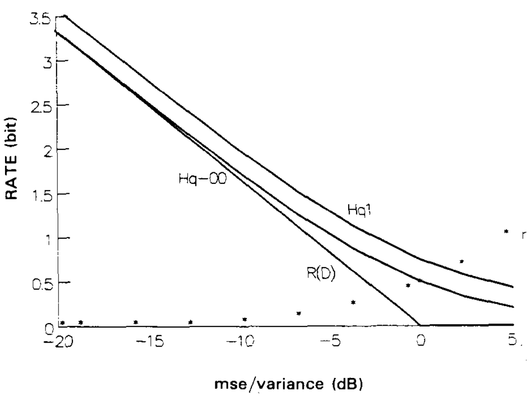

Fig. 3. Rate of the randomized lattice quantizer for a memoryless Gaussian source, as compared to the rate of the uniform randomized quantizer.

the rate-distortion function was only the resolution measure $(1/2)\log(P(Y)/P(X))$. As a matter of fact, using a technique similar to the derivation in Appendix C, this resolution measure converges as $K \to \infty$ to $(1/2)\log(1 + \epsilon/\sigma_x^2)$ where σ_x^2 is the variance of the Gaussian source, and this is the redundancy illustrated in Fig. 3.

In both Fig. 2 and Fig. 3, we have indicated (with *) the resolution measure of the Gaussian random variable.

Calculating explicitly the rate distortion function can be complicated for non Gaussian sources. However, the resolution measure that appears in (36) for the difference between the rate of the randomized quantizer and the rate-distortion function can be calculated relatively easy. In Fig. 4 we present this measure as a function of the quantizer step-size (normalized by the source variance, σ), for the uniform memoryless source (U), the memoryless Laplacian source (L) and the memoryless Gaussian source (G). Note that since the low resolution case of SNR = 1 correspond to $\Delta/\sigma = \sqrt{12}$ the plots in Fig. 4 correspond to high and a medium resolution cases.

IV. Summary and Conclusion

We have provided an expression and found universal upper bounds for the difference between the rate of the randomized uniform and lattice quantizers and the rate distortion function for any given source. This difference, defined as the redundancy of the quantizer, was upper bounded by two procedures. In the first procedure, we achieved a bound which hold for all distortion levels and for all sources, and can be described as the capacity of the channel of Fig. 1; however, this bound is tight only for low resolution quantization. The second procedure leads to a bound that depends on the allowable distortion; this bound becomes universal and converges to the results obtained for deterministic uniform/lattice quantizers, at the high resolution, or low distortion, region.

Our results assume that the dither samples are drawn independently. When the dither is drawn only once, as suggested in [1], the rate of the quantizer can be slightly different. However, it is easy to see that the constant universal bound (the capacity bound), and our expression for the rate at high resolution hold in this case too. An expression for the rate, in general, of this alternative procedure of dithering has been calculated, and can be found in [8].

The uniform and lattice quantizers are much simpler than the optimal vector quantizers, which can be found by the procedures of [9] and [3]. There is no need to predesign these universal quantizers, and one can use efficient algorithms, e.g., [10], for decoding the lattice code. One important consequence of our result is that the degradation in performance can be assessed. While the bound presented in [1] is not useful in many practical application since it is too far away from the optimal performance, our expression and bounds can be practically important in a wide region of allowable distortion levels for deciding whether to choose the simple lattice quantizer over the more complicated ECVQ.

Appendix A
Generalization of Theorem 1

Let us first assume that the input random vector X is of dimension K, as the dimension of the lattice. We will show that the entropy of the randomized quantizer is given by

$$H(Q_K \mid Z_K) = I(X_K; Y_K) = I(X_K; X_K + N_K)$$
$$= H(Y_K) - \log V, \qquad (45)$$

where N_K is uniformly distributed over $\mathscr{V}_0$ and the subscript K refers to the fact that the vectors are K-dimensional. The probability density function of the random vector $Y_K = X_K + N_K$ is given by

$$f_{Y_K}(y) = f_{X_K}(x) * f_{N_K}(n) = \int_{R^K} f_{X_K}(x) f_{N_K}(y-x)\, dx$$
$$= \frac{1}{V}\int_{\mathscr{V}_0 + y} f_{X_K}(x)\, dx. \qquad (46)$$

By definition,

$$H(Q_K \mid Z_K) = H(Q_K(X_K + Z_K) \mid Z_K)$$
$$= -E\{\log P(Q_K(x+z) \mid z)\},$$

where the expectation is with respect to X_K and Z_K. This is the average of discrete entropies since $Q_K(\cdot)$ can take the values $\{q_i\}$. Thus we can write

$$H(Q_K \mid Z_K) = E_{Z_K}\Big\{-\sum_i P(q_i \mid z_K) \log P(q_i \mid z_K)\Big\}. \qquad (47)$$

The region $\mathscr{V}_i$ is the Voronoi region associated with the ith quantization point, i.e., $\mathscr{V}_i = \mathscr{V}_0 + q_i$. The probability of a quantization point q_i, given the dither value z_K, is

$$P(q_i \mid z_K) = \text{Prob}\{x_K + z_K \in \mathscr{V}_i\}$$
$$= \text{Prob}\{x_K + z_K \in q_i + \mathscr{V}_0\}$$
$$= \int_{q_i + \mathscr{V}_0 - z_K} f_{X_K}(\alpha)\, d\alpha = V \cdot f_{Y_K}(q_i - z_K), \qquad (48)$$

where the last equality was achieved using (46). Substituting (48) in (47) leads to

$$H(Q_K \mid Z_K) = E_{Z_K}\Big\{-\sum_i V \cdot f_{Y_K}(q_i - z_K) \log V \cdot f_{Y_K}(q_i - z_K)\Big\}.$$

Now, since the dither is uniformly distributed over $\mathscr{V}_0$, we get

$$H(Q_K \mid Z_K) = \frac{1}{V}\int_{\mathscr{V}_0} dz\Big[\sum_i V \cdot f_{Y_K}(q_i - z) \log V \cdot f_{Y_K}(q_i - z)\Big]$$
$$= \int_{R^K} f_{Y_K}(y) \log\left(V \cdot f_{Y_K}(y)\right) dy$$
$$= H(Y_K) - \log V,$$

since summation over the lattice points and integration over the lattice Voronoi cell complete each other into integration over $\mathscr{R}^K$; thus, (45) is proven.

To get the more general result, (20), we have to follow the same steps as above where now instead of integrating over $\mathscr{V}_i$ we integrate over the n/K-fold Cartesian product of these Voronoi cells and normalize the result to get the entropy per single sample. Thus, if these subvectors are i.i.d., $H(Y) = (n/K) \cdot H(Y_K)$, $I(X;Y) = (n/K) \cdot I(X_K; Y_K)$, etc. The result (20), however, holds also when the subvectors are dependent.

Appendix B
The Capacity of the Channel of Figure 1

In the appendix, we derive upper and lower bounds for the capacity of the channel of Fig. 1. We will present in details the scalar case. The extension to the vector case is not very complicated and it has been mentioned, in a partial way, in the body of the paper.

In the scalar case, the capacity we are looking for is

$$C = \max_{\{f_X(x):\, E\{\delta(X)\} \le \epsilon\}} I(X;Y)$$
$$= \max_{\{f_X(x):\, E\{\delta(X)\} \le \epsilon\}} H(Y) - \log \Delta, \qquad (49)$$

where $Y = X + N$, $N \sim \mathscr{U}(-\Delta/2, \Delta/2)$ and

$$\epsilon = E\{\delta(N)\} = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} \delta(n)\, dn = \phi(\Delta).$$

A lower bound for this capacity is readily achieved using the entropy-power inequality

$$P(Y) = P(X+N) \ge P(X) + P(N),$$

where $P(\cdot)$ is the entropy power of the corresponding random variable, e.g., $P(X) = (1/2\pi e)2^{2H(X)}$. The tightest lower bound achieved this way will use the input X^* that maximizes the entropy under the constraint $E\{\delta(X)\} \le \epsilon$; we denote its entropy $H(X^*)$ and we get the lower bound

$$C \ge \frac{1}{2}\log\left(2^{2H(X^*)} + 2^{2H(N)}\right) - H(N)$$
$$= \frac{1}{2}\log \frac{2^{2H(X^*)} + 2^{2H(N)}}{2^{2H(N)}}. \qquad (50)$$

Upper bounds for this capacity will be achieved by stating constraints on the output random variable and substituting in (49) for $H(Y)$ the entropy $H(Y^*)$ where Y^* is the r.v. that maximizes the entropy under these constraints.

We now demonstrate these bounding techniques for the square error distortion measure. In this case $E\{X^2\} \le \epsilon$ and $E\{N^2\} = \epsilon = \Delta^2/12$. Thus, $E\{Y^2\} \le 2\epsilon$ and we have a second moment constraint on the output. Under this constraint the output that maximizes the entropy is Gaussian with entropy $H(Y^*) = 1/2 \log 2\pi e \cdot 2\epsilon$. So , the capacity is upper bounded by

$$C \le \frac{1}{2}\log 2\pi e \cdot 2\epsilon - \frac{1}{2}\log \Delta^2 = \frac{1}{2}\log\frac{4\pi e}{12}$$
$$\approx 0.754 \text{ b/sample}. \qquad (51)$$

The input to the channel also satisfies a second moment constraint $E\{X^2\} \le \epsilon$, so X^* is also Gaussian with entropy $1/2 \log 2\pi e \cdot \epsilon$. Thus, the lower bound (50) becomes

$$C \ge \frac{1}{2}\log\frac{2\pi e\epsilon + 12\epsilon}{12\epsilon} = \frac{1}{2}\log\left(1 + \frac{2\pi e}{12\epsilon}\right)$$
$$\approx 0.638 \text{ b/sample}, \qquad (52)$$

where, of course, we substituted $2^{2H(N)} = \Delta^2 = 12\epsilon$.

When we generalize to the lattice quantizer case we have to replace $G_1 = 1/12$ by G_K and get (33).

Consider now distortion measure of the form $|x|^r$. In this case $E\{|X|^r\} \le \epsilon$ and $E\{|N|^r\} = \epsilon = \Delta^r/(r+1)2^r$. The noise entropy can be written in this case as

$$H(N) = \log \Delta = \frac{1}{r} \log \left[(r+1)2^r \cdot \epsilon \right]$$

or $2^{2H(N)} = \Delta^2 = 4[(r+1)\epsilon]^{2/r}$.

The lower bound is easily found by substituting the source X^* that maximizes the entropy under the rth moment constraint. The entropy of such a source is given by

$$H^r(S^*) = \frac{1}{r} \log \left[e \cdot 2^r \cdot \Gamma^r\left(1 + \frac{1}{r}\right) \cdot r\epsilon \right]. \quad (53)$$

Thus, the channel capacity is lower bounded by

$$C \ge \frac{1}{2} \log \left[1 + \Gamma^2\left(1 + \frac{1}{r}\right) \cdot \left(\frac{e \cdot r}{r+1}\right)^{2/r} \right] = C_l, \quad (54)$$

which takes the values $0.755, 0.638, 0.595, \cdots, 0.5$ for $r = 1, 2, 3, \cdots, \infty$.

The upper bounding technique is slightly more complicated and the bound we get may be loose since we cannot easily get moments constraints on the output random variable Y. We can only bound the rth moment using Holder's inequality,

$$E\{|Y|^r\} \le E\{(|S| + |N|)^r\} \le \epsilon \cdot \sum_{k=0}^{r} \binom{r}{k} \frac{(r+1)^{k/r}}{k+1} \le \epsilon \cdot 2^r. \quad (55)$$

The maximum entropy of the output under this rth moment constraint is given by (53) where we substitute $2^r\epsilon$ for ϵ. Thus we get the upper bound

$$C \le \frac{1}{r} \log \left[e \cdot 2^r \cdot \Gamma^r\left(1 + \frac{1}{r}\right) \cdot \frac{r}{r+1} \right] = C_u, \quad (56)$$

where C_u takes the values $1.44, 1.254, 1.180, \cdots, 1$ for $r = 1, 2, 3, \cdots, \infty$. We see immediately that this upper bound is loose at least for $r = 2$. It can be tightened for even r, i.e., $r = 2p$. In this case, we can use the fact that the odd moments of N are zero and get

$$E\{Y^{2p}\} \le \epsilon \cdot \sum_{k=0}^{p} \binom{2p}{2k} \frac{(2p+1)^{k/p}}{2k+1} \le \epsilon \cdot 2^{2p-1}, \quad (57)$$

which improves (56) by $1/r$ and so we get

$$C \le \frac{1}{r} \log \left[e \cdot 2^{r-1} \cdot \Gamma^r\left(1 + \frac{1}{r}\right) \cdot \frac{r}{r+1} \right] = C_u, \quad r = 2p. \quad (58)$$

Note that C_u now takes the values $0.754, 0.888, \cdots, 1$ for $r = 2, 4, \cdots, \infty$.

Similar results have been obtained in [2] for the bounds there.

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A Note on the Competitive Optimality of the Huffman Code

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Abstract—**It is known that a bound on the probability that the length of any source code will be shorter than the self information by γ bits can be obtained using a Chebychev-type argument. From this bound, one can establish the competitive optimality of the self information and of the Shannon-Fano code (up to one bit). In general, however, the Huffman code cannot be examined using this technique. Nevertheless, in this correspondence the competitive optimality (up to one bit) of the Huffman code for general sources is also established using a different technique.**

Index Terms—**Competitive optimality, Huffman code, self information, Chebychev inequality.**

I. Introduction

Given the probability of a source one can design a uniquely decodable (UD) source code that minimizes the expected codelength. This expected code length must be, of course, greater than the entropy of the source. The optimal code in this sense would assign to each outcome x a codeword of length $-\log p(x)$, the self information, and its expected length would exactly be the entropy. (Throughout the correspondence $\log x = \log_2 x$.) However, the self information may not be an integer. Incorporating the Diophantine constraints, it is well known that the Huffman code minimizes the expected codelength.

Manuscript received August 7, 1990; revised May 24, 1991. This work was supported by The Andrew W. Mellon Foundation, Woods Hole Oceanographic Institution.

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IEEE Log Number 9103963.